# MA30253: Continuum Mechanics

In this paper the domain $\Omega \subset \mathbb{R}^3$ denotes the initial configuration of a continuum, We denote by $\boldsymbol{\phi} = \boldsymbol{\phi}(\mathbf{X}, t)$, $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$ a motion of the continuum and $\Omega_t = \boldsymbol{\phi}(\Omega, t)$ denotes the image of the initial configuration at time $t$ (with $\Omega_0 = \Omega$).

$\mathbf{X} = (X_1, X_2, X_3)^T$ denote material coordinates in $\Omega$ and $\mathbf{x} = (x_1, x_2, x_3)^T$ denote spatial coordinates in $\Omega_t$.

The Euler equations for the flow of an inviscid, incompressible fluid of constant density $\rho_0$ are given by:

$$\rho_0 \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla P + \rho_0 \mathbf{b} \ , \ \mathbf{x} \in \Omega_t \ , \tag{1}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_i(\mathbf{x}, t))$ is the (spatial) velocity, $P(\mathbf{x}, t)$ is the pressure and $\mathbf{b}(\mathbf{x}, t)$ is the body force per unit mass.

The conservation of mass condition then reduces to the incompressibility condition

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = 0 \ \text{ in } \Omega_t. \tag{2}$$

We denote the material time derivative by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \ .$$

1. Consider a planar/laminar steady flow in $\mathbb{R}^3$ in which the spatial velocity field is given by

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \\ 0 \end{pmatrix}$$

is a solution of the Euler equations (1).

(i) Suppose further that the flow is incompressible and irrotational and derive the conditions satisfied by the scalar functions $u$ and $v$.

(ii) Define the corresponding complex velocity ~~potential~~ $w(z)$ where $z \in \mathbb{C}$ and, assuming that $u, v$ are continously differentiable, show that $w(z)$ defines an analytic function on its domain of definition.

(iii) What is the corresponding complex velocity potential $\Phi(z)$ and stream function $\psi(x_1, x_2)$? Show that the stream function is constant along streamlines of the flow.

(iv) Show that the real part of the complex contour integral

$$\oint_\gamma w(z)\, dz$$

around a contour $\gamma$, parametrised as $z(s) = x(s) + iy(s),\ \ s \in [a, b]$, is the circulation of the flow around the contour.

(v) State Blasius Theorem for the force exerted by a flow on a body $B$ immersed in the flow. Use it to calculate the force on the unit disc (centre the origin) in the case when the complex velocity potential is given by

$$\Phi(z) = z + \frac{1}{z}.$$

Q2. Let $\boldsymbol{\phi}(\mathbf{X}, t)$, $\boldsymbol{\phi} : \Omega \times [0, \infty) \to \mathbb{R}^3$ be a motion of a continuum. Let $\rho_0(\mathbf{X})$ be the density in the initial configuration at $\mathbf{X} \in \Omega$ and $\rho(\mathbf{x}, t)$ the density in the configuration at time $t$ at $\mathbf{x} \in \Omega_t$, where $\Omega_t = \boldsymbol{\phi}(\Omega, t)$ (and $\Omega = \Omega_0$).

(a) Define the corresponding spatial velocity field $\mathbf{v}(\mathbf{x}, t) = (v_i(\mathbf{x}, t))$ for $\mathbf{x} \in \Omega_t$.

(b) Show that it follows from conservation of mass that

$$\rho(\mathbf{x}, t) = \frac{\rho_0(\mathbf{X})}{\det D\boldsymbol{\phi}(\mathbf{X}, t)}, \quad \text{with } \mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t),$$

where $D\boldsymbol{\phi}$ denotes the material gradient matrix $F(\mathbf{X}, t) = \left(\frac{\partial \phi_i(\mathbf{X},t)}{\partial X_j}\right)$.

(c) Prove that for any spatially defined function $f(\mathbf{x}, t)$, we have

$$\frac{d}{dt}\left(\int_{B_t} \rho(\mathbf{x}, t) f(\mathbf{x}, t)\right) dV = \int_{B_t} \rho(\mathbf{x}, t) \frac{D}{Dt} f(\mathbf{x}, t)\, dV$$

for any subdomain $B_t \subset \Omega_t$.

(d) Prove that the material gradient matrix $F$ satisfies

$$\frac{\partial}{\partial t} F(\mathbf{X}, t) = \Gamma(\mathbf{x}, t) F(\mathbf{X}, t), \ \ \mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t),$$

where $\Gamma(\mathbf{x}, t) = (\Gamma_{ij}) = \left(\frac{\partial v_i}{\partial x_j}\right)$ denotes the spatial velocity gradient.

(e) Prove that

$$\frac{\partial}{\partial t}(\det(D\boldsymbol{\phi}(\mathbf{X}, t)) = \det(D\boldsymbol{\phi}\,(\mathbf{X}, t))\ (\nabla.\mathbf{v}(\mathbf{x}, t)), \ \ \mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t),$$

and hence that an incompressible flow must correspond to a motion satisfying $\det D\boldsymbol{\phi}(\mathbf{X}, t) \equiv 1$.

Q3. Consider an incompressible ideal fluid subject to a motion $\phi : \Omega \times [0, T] \to \mathbb{R}^3$, with corresponding (spatial) velocity field $\mathbf{v}(\mathbf{x}, t) = (v_i(\mathbf{x}, t))$ and constant density $\rho_0$ satisfying (1), (2).

(a) The corresponding vorticity is defined by $\boldsymbol{\omega}(\mathbf{x}, \mathbf{t}) = \nabla \times \mathbf{v}(\mathbf{x}, \mathbf{t})$ for $\mathbf{x} \in \Omega_t = \phi(\Omega, t)$.

(i) What is a vortex line?

(ii) Prove that vortex lines are transported by the flow.

(You may assume Cauchy's result that $\boldsymbol{\omega}(\mathbf{x}, t)|_{\mathbf{x}=\phi(\mathbf{X},t)} = D\phi(\mathbf{X}, t)\boldsymbol{\omega}_0(\mathbf{X})$ for all $\mathbf{X} \in \Omega$, where $\boldsymbol{\omega}_0(\mathbf{X})$ denotes the vorticity in the initial configuration at $\mathbf{X} \in \Omega$.)

(b) Define the corresponding spin tensor $W$ and rate of stretch tensor $S$. Prove that if $\mathbf{v}(\mathbf{x}, t)$ satisfies (1) and (2), then

(i) $tr\, S = 0$,

(ii) $\omega_i = \epsilon_{ijk} W_{kj}$ (where $\epsilon_{ijk}$ denotes the alternating symbol),

(iii) $2W_{ij} = \epsilon_{ikj}\omega_k$ .

(iv) Suppose further that the spatial velocity gradient tensor $\Gamma(\mathbf{x}, t) = (\Gamma_{ij}) = \left(\frac{\partial v_i}{\partial x_j}\right)$ is skew symmetric. Prove that

$$\frac{DW}{Dt} = 0 .$$

(You may assume that the vorticity satisfies $\frac{D\boldsymbol{\omega}}{Dt} = \Gamma\boldsymbol{\omega}$. )

4. Consider an incompressible ideal fluid subject to a motion $\phi : \Omega \times [0,T] \to \mathbb{R}^3$, with corresponding (spatial) velocity field $\mathbf{v}(\mathbf{x},t) = (v_i(\mathbf{x},t))$ and constant density $\rho_0$.

(a) Prove the vector identity

$$(\mathbf{v}.\nabla)\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \frac{1}{2}\nabla||\mathbf{v}||^2, \text{ where } \boldsymbol{\omega} = \nabla \times \mathbf{v} \text{ is the vorticity .}$$

(b) Show that the spin tensor $W = \frac{1}{2}\left[\Gamma - \Gamma^T\right]$ satisfies $2W\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^3$, where $\omega_j = \epsilon_{jmn} W_{nm}$, $\Gamma(\mathbf{x},t) = (\Gamma_{ij}(\mathbf{x},t)) = \left(\frac{\partial v_i(\mathbf{x},t)}{\partial x_j}\right)$ is the velocity gradient tensor and $\epsilon_{jmn}$ is the alternating (permutation) symbol.

(c) By taking the curl of the linear momentum equations (1), show that

$$\frac{D\boldsymbol{\omega}}{Dt} = \Gamma\boldsymbol{\omega}.$$

(You may assume that $(\nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = (\mathbf{v}.\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}.\nabla)\mathbf{v}.$ )

(d) Show that

$$\boldsymbol{\omega}(\phi(\mathbf{X},t),t) = F(\mathbf{X},t)\boldsymbol{\omega}(\mathbf{X},0) \quad \text{for } \mathbf{X} \in \Omega$$

satisfies the equations in part (c), where

$$F(\mathbf{X},t) = (F_{ij}(\mathbf{X},t)) = \left(\frac{\partial \phi_i}{\partial X_j}\right)$$

is the deformation gradient tensor.

(You may assume that $\frac{\partial F(\mathbf{X},t)}{\partial t} = \Gamma(\mathbf{x},t)F(\mathbf{X},t), \quad \mathbf{x} = \phi(\mathbf{X},t)$.)

(e) In the case that $\Gamma$ is skew symmetric, prove that the material derivative of the vorticity is zero. Interpret this result.

5. In this question, $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ are cartesian bases for $\mathbb{R}^3$ related by $\tilde{\mathbf{e}}_i = q_{ij}\mathbf{e}_j$ where $Q = (q_{ij}) \in SO(3)$. There is a corresponding change of cartesian coordinates $X_i \to \tilde{X}_i = q_{ik}X_k$.

(a) (i) What is meant by saying that $T_{i_1 \ldots i_n}$, where $i_1, \ldots, i_n$ can take the values $1, 2, 3$, are the components of a *cartesian tensor* $\underline{\underline{T}}$ *of order* $n$ (i.e., a CTn) in the basis $\mathcal{A}$?

(ii) What is meant by saying that $T_{i_1 \ldots i_n}$ are components of an *isotropic* CTn?

(iii) Define the *alternating symbol* $\epsilon_{ijk}$.

(b) Suppose that $S_{ijk} = \epsilon_{ijk}$, $i, j, k = 1, 2, 3$. Show that $S_{ijk}$ are the components of an isotropic CT3.

(c) Show that the triple scalar product, defined by

$$\langle \mathbf{X}, \mathbf{Y} \times \mathbf{Z} \rangle = \epsilon_{ijk} X_i Y_j Z_k = \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix},$$

where $\mathbf{X} = X_i\mathbf{e}_i, \mathbf{Y} = Y_i\mathbf{e}_i, \mathbf{Z} = Z_i\mathbf{e}_i \in \mathbb{R}^3$, defines a CT0.

(d) Suppose that $S_i$, $i = 1, 2, 3$, have the property that $S_iT_i$ is a CT0 for any cartesian tensor $\underline{\underline{T}}$ of order 1 with components $T_i$. Prove that $S_i$ are the components of a CT1.

Hence prove that $(\mathbf{Y} \times \mathbf{Z})_i = \epsilon_{ijk}Y_jZ_k$ are components of a CT1 for any $\mathbf{Y}, \mathbf{Z} \in \mathbb{R}^3$.

6. In this question, a change of cartesian coordinates from $\mathbf{X} = (X_1, X_2, X_3)^T$ to $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)^T$ corresponds to $X_i \to \tilde{X}_i = q_{ik}X_k$ for some $Q = (q_{ij}) \in SO(3)$.

(a) What is meant by saying that $T_{i_1 \ldots i_n}$, where the indices $i_1, \ldots, i_n$ can take the values $1, 2, 3$, are the *components of a* CTn (i.e., of *a cartesian tensor of order n*)?

(b) Define what is meant by *contraction* of the components of a CTn. Show that contraction of the components of a CTn gives the components of a CT(n-2).

(c) Suppose that $T_{i_1 \ldots i_n}$ are functions of $X_1, X_2, X_3$ and are components of a CTn. Show that

$$S_{i_1 \ldots i_n k} = \frac{\partial}{\partial X_k} T_{i_1 \ldots i_n}$$

define the components of a CT(n+1).

(d) Let $\phi(X_1, X_2, X_3)$ be a scalar-valued function. Prove that the Laplacian of $\phi$,

$$\Delta\phi = \frac{\partial^2 \phi}{\partial {X_1}^2} + \frac{\partial^2 \phi}{\partial {X_2}^2} + \frac{\partial^2 \phi}{\partial {X_3}^2},$$

defines a CT0.

(e) Suppose that $S_{ijk}$, $i, j, k = 1, 2, 3$, have the property that for any CT1 with components $T_i$,

$$U_{ij} = S_{ijk} T_k$$

are the elements of a CT2. Prove that $S_{ijk}$ are the elements of a CT3.

7. Let $\mathbf{v}(\mathbf{x},t) = (v_i(\mathbf{x},t))$ be a given spatial velocity field.

(a) Define the corresponding *rate of stretch tensor* $S$ and *spin tensor* $W$ and show that $\Gamma = S + W$, where $\Gamma = (\Gamma_{ij})$, $\Gamma_{ij} = \frac{\partial v_i}{\partial x_j}$ is the velocity gradient tensor.

(b) If $\mathbf{v}(\mathbf{x},t)$ satisfies the Euler equations, show, by differentiating (1) with respect to $x_j$, that the corresponding velocity gradient tensor $\Gamma$ satisfies

$$\frac{D\Gamma_{ij}}{Dt} + \left(\Gamma^2\right)_{ij} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}$$

Hence prove that the rate of stretch tensor $S$ and spin tensor $W$ satisfy

$$\frac{DS_{ij}}{Dt} + (S^2 + W^2)_{ij} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}.$$

and

$$\frac{DW_{ij}}{Dt} + (SW + WS)_{ij} = 0$$

(c) If $S_0, W_0$ are constant tensors, with $S_0$ symmetric and $W_0$ skew-symmetric, show that

$$\mathbf{v}(\mathbf{x}) = S_0\mathbf{x} + W_0\mathbf{x}$$

gives rise to a solution of the equations in part (b) provided that $S_0W_0 + W_0S_0 = 0$. Find the corresponding pressure.

(d) In the case $v_i(\mathbf{x},t) = \frac{1}{2}\epsilon_{ijk}b_jx_k$, where $\mathbf{b} = (b_i) \in \mathbb{R}^3$, calculate the corresponding $S_0$ and $W_0$ and describe the associated motion.
(You may assume that if $W \in M^{3\times 3}$ is skew-symmetric, then $Q(t) = e^{Wt}$, $t \in \mathbb{R}$ satisfies $Q(t) \in SO(3)$ for all $t$ and $Q(0) = I$.)